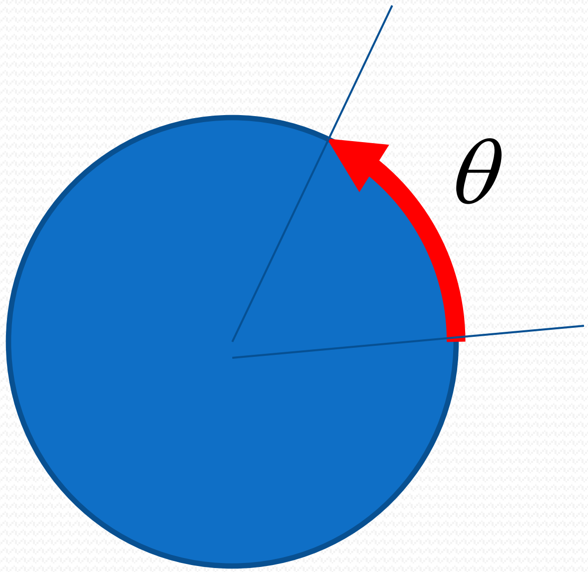
Chapter 10

Rotation of a Rigid Object About a Fixed Axis

Angular Position, Velocity, and Acceleration

tricky part: θ=0 and θ=2π are the same

- Main change: position → angle
- Can introduce [angular] displacement, velocity, acceleration



$$\Delta\theta = \theta_f - \theta_i$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Constant Angular Acceleration

- Keep all formulas from constant linear acceleration

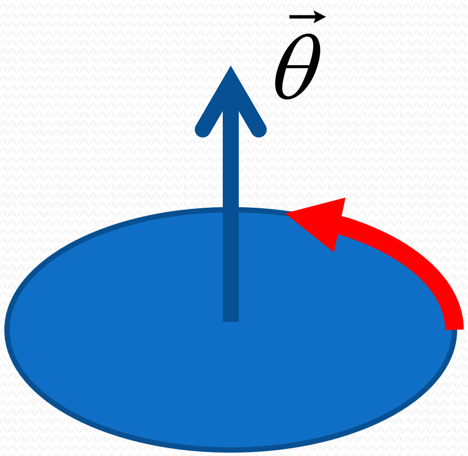
$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

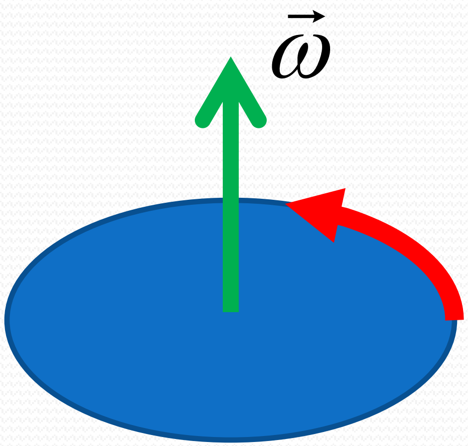
$$\omega_f = \omega_i + \alpha t$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

Angular Position, Velocity, and Acceleration as Vectors

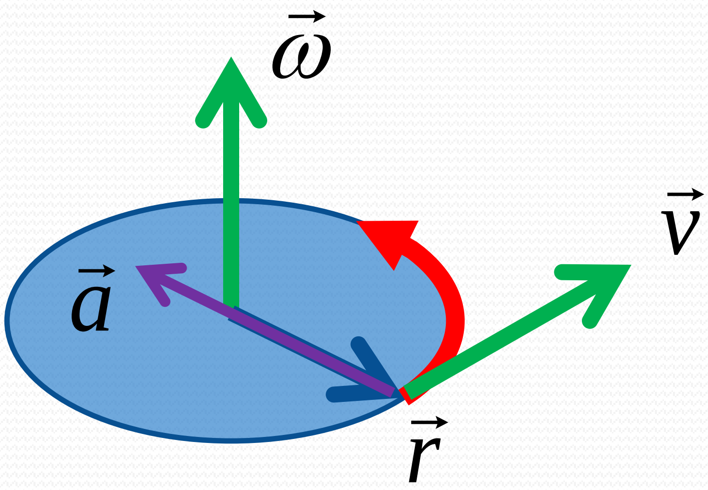
- Directed along the rotation axis – use right-hand screw rule to figure out the direction





Angular and Translational Quantities

- When a rigid object rotates, each particle in it undergoes accelerated motion



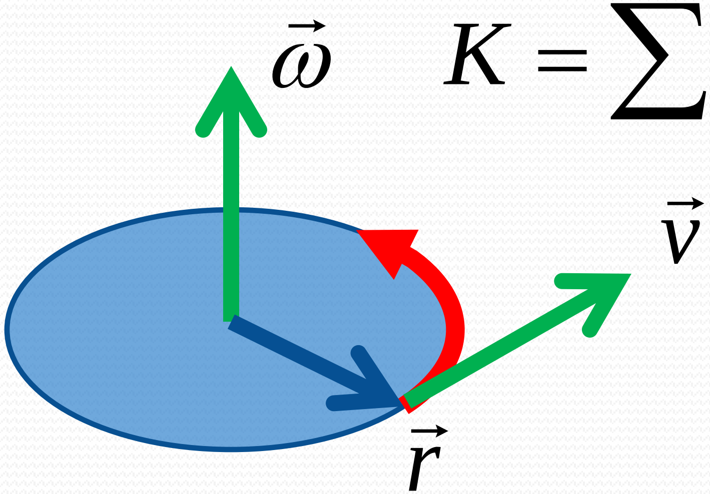
$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$a = \frac{v^2}{r} = r\omega^2$$

uniform circular motion

Rotational Kinetic Energy

- Since components of a rotating rigid object move, there is kinetic energy associated with this motion



$$K = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} I \omega^2$$

same for all points

$$I = \sum m_i r_i^2$$

discrete case

$$I = \int r^2 dm$$

continuous case

moment (not momentum!) of inertia

Mass, Center of Mass, and Moment of Inertia

$$M = \sum m_i \qquad \vec{r}_{CM} = \frac{1}{M}\sum m_i \vec{r}_i \qquad I = \sum m_i r_i^2$$

$$M = \int dm \qquad \vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm \qquad I = \int r^2 dm$$

$$M = \int \rho dV \qquad \vec{r}_{CM} = \frac{1}{M}\int \rho \vec{r}\, dV \qquad I = \int \rho r^2 dV$$

Warning: r^2 here is NOT equal to $x^2+y^2+z^2$!

Moments of inertia

- We define moments about an axis
- r is a distance to the axis ($r^2 \neq x^2+y^2+z^2$)

$$I_x = \int \rho(y^2 + z^2)dV$$

$$I_y = \int \rho(z^2 + x^2)dV$$

$$I_z = \int \rho(x^2 + y^2)dV$$

If the object is not symmetric w.r.t. given axis, the I calculation can become involved!

for a flat object in x-y plane (z=0), $I_x+I_y=I_z$ (perpendicular axis theorem)

Parallel Axis Theorem

- In general, the moment of inertia depends on axis of rotation – have to re-calculate it for each axis
- Parallel axis theorem: if we know the moment of inertia about an axis passing through the center of mass I_{CM}, for an arbitrary parallel axis shifted by distance D,

$$I = I_{CM} + MD^2$$

Proof: assuming CM is at 0, $I_z = \int\left((x-D)^2 + y^2\right) dm =$

$$\int (x^2 + y^2)\, dm + D^2 \int dm - 2D \int x\, dm = I_{z\,CM} + MD^2$$

=0, since it's CM

Torque

- Torque is to rotation what force is to linear motion

$$v = \frac{dx}{dt} \longrightarrow \omega = \frac{d\theta}{dt}$$

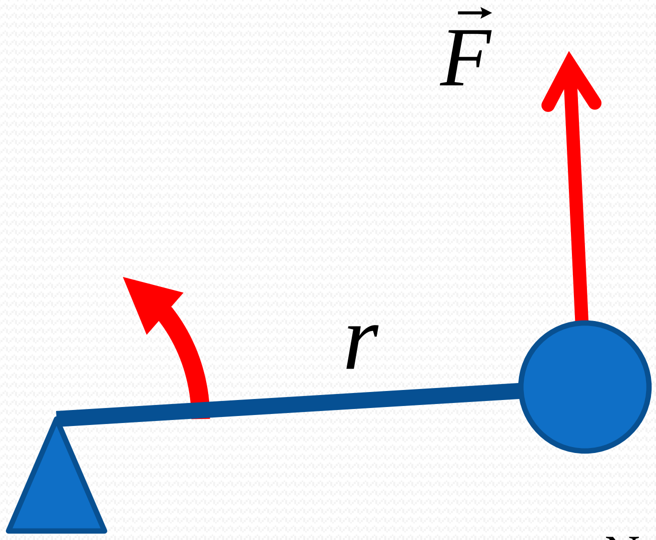
$$a = \frac{dv}{dt} \longrightarrow \alpha = \frac{d\omega}{dt}$$

$$K = \frac{1}{2}mv^2 \longrightarrow K = \frac{1}{2}I\omega^2$$

$$F = ma \longrightarrow \tau = I\alpha$$

Torque and Force

$$\tau = Fr$$



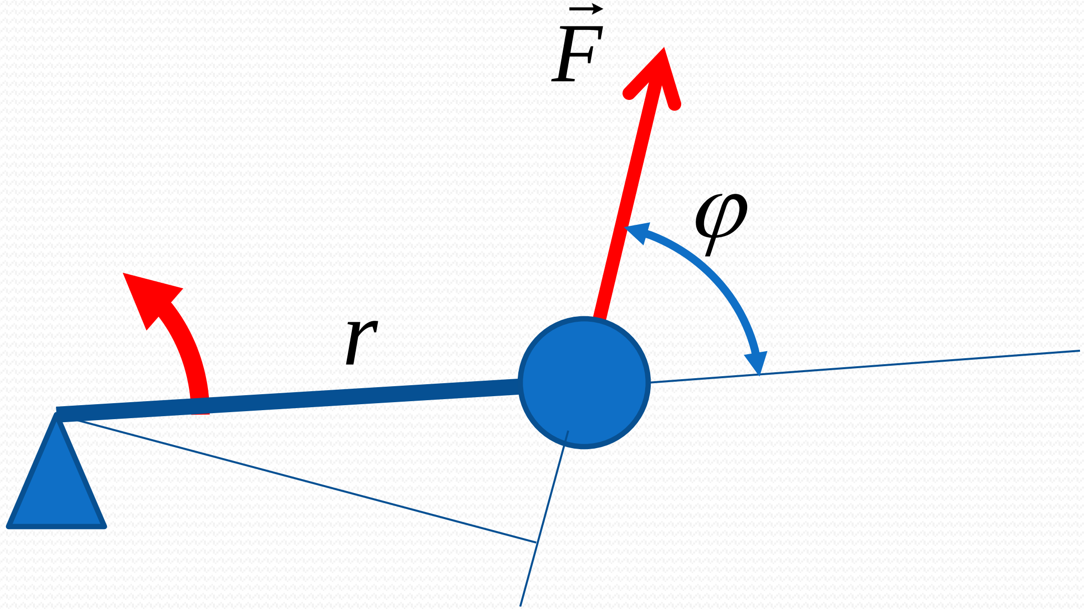
$$F = ma = m\alpha r$$

$$\tau = Fr = m\alpha r^2 = I\alpha$$

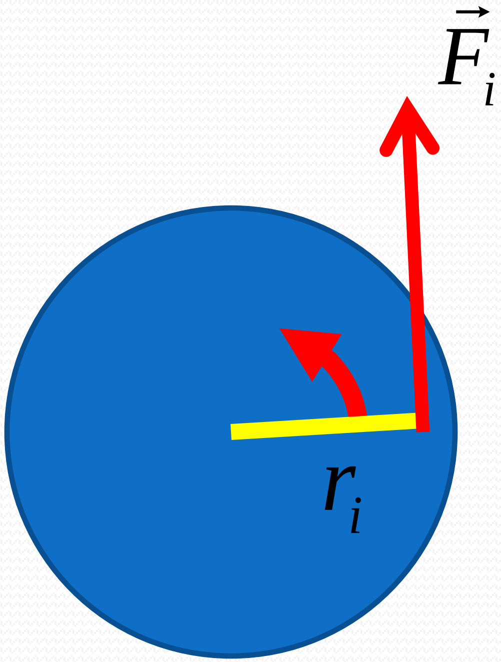
Note: torque formula looks similar to work, but the two are completely different things!

Torque and Force

$$\tau = Fr\sin\varphi$$



Torque and Force



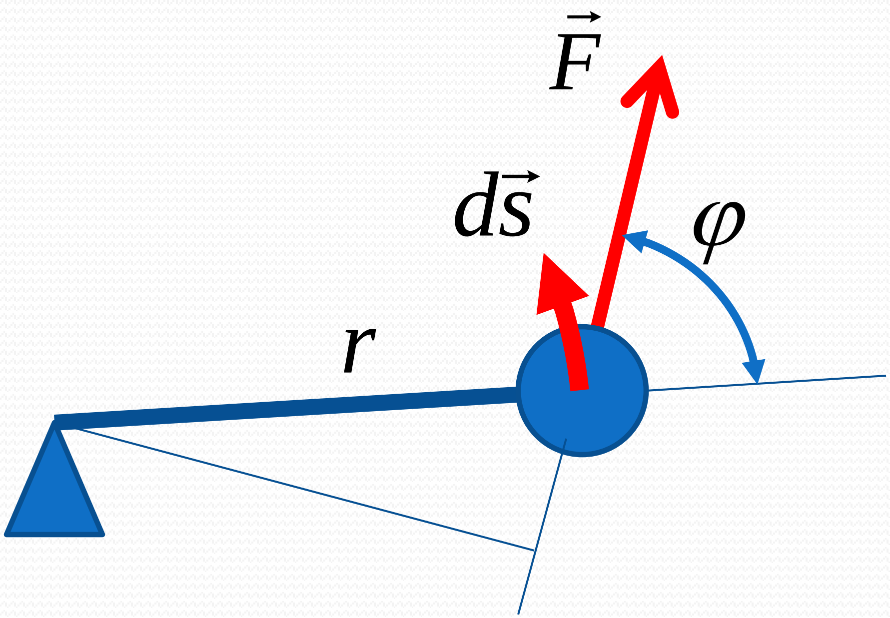
$$F_i = m_i a_i = m_i \alpha r_i$$

$$\tau_i = F_i r_i = m_i \alpha r_i^2$$

$$\tau = \sum_i \tau_i = \alpha \sum_i m_i r_i^2 = I\alpha$$

Work Done by a Torque

$$dW = \vec{F}\, d\vec{s} = F\, ds \sin\varphi = Fr\, d\theta \sin\varphi = \tau\, d\theta$$



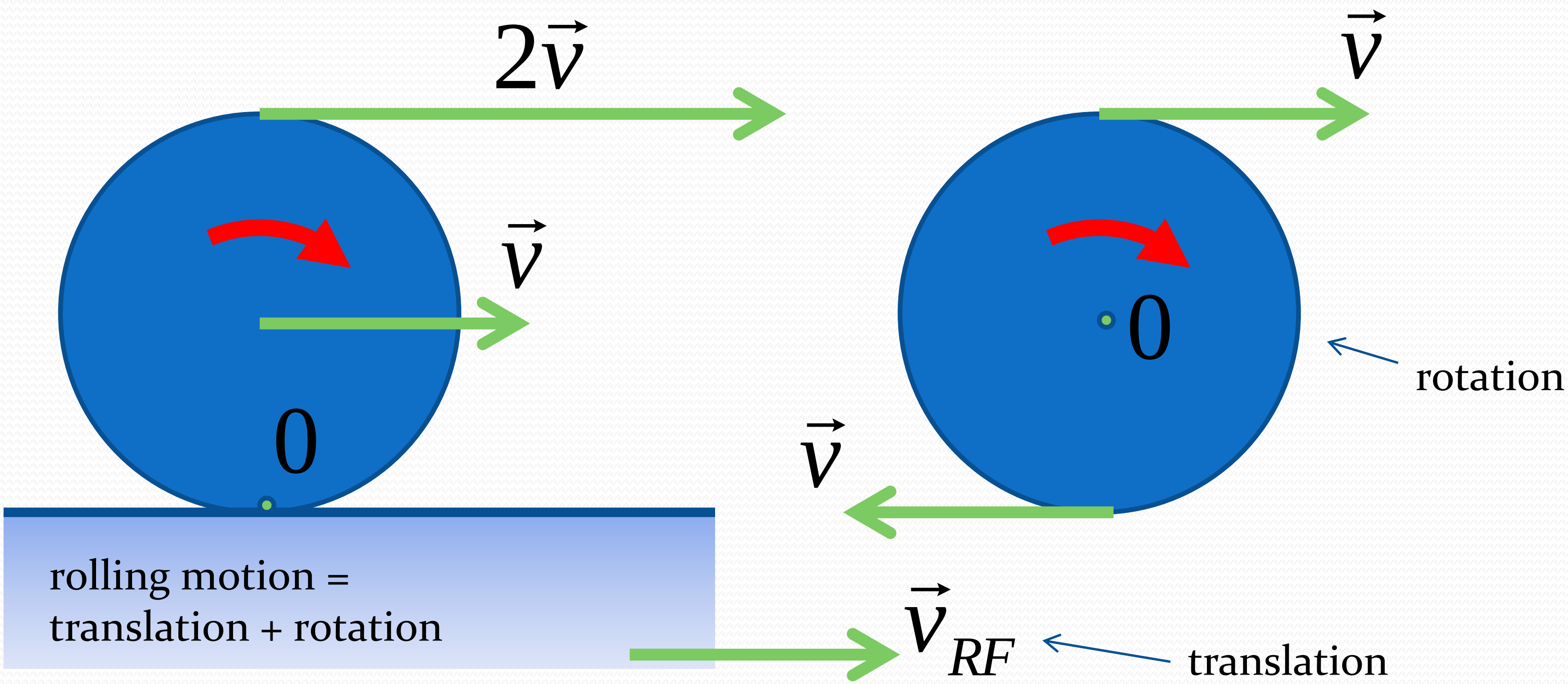
$$\tau = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

$$dW = I\omega\frac{d\omega}{d\theta}d\theta = I\omega\, d\omega$$

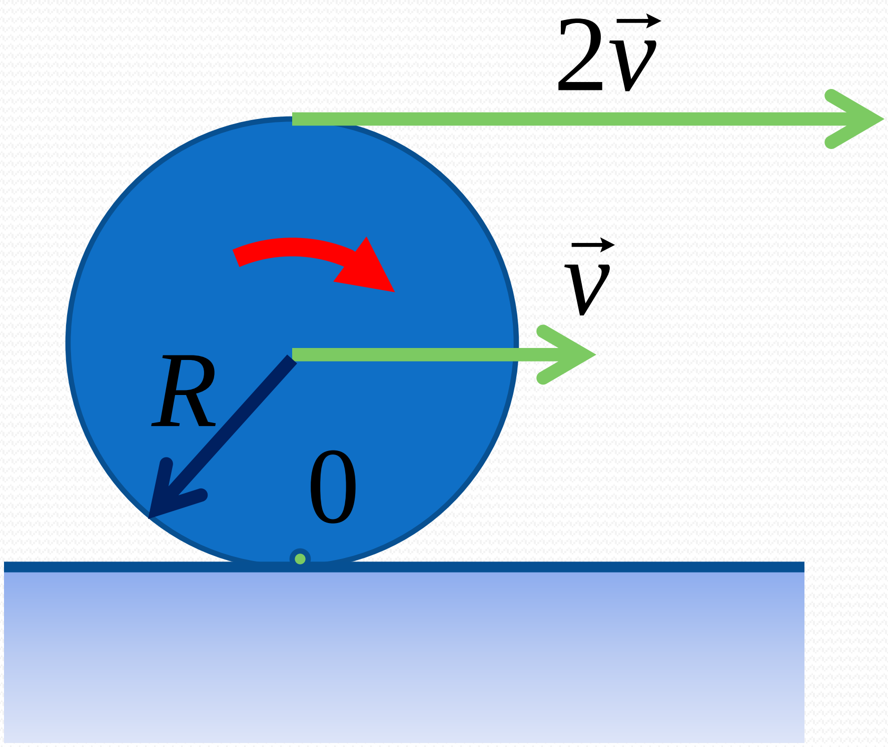
$$W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = K_f - K_i$$

Rolling Rigid Object

- Rolling without slipping (a.k.a. pure rolling motion): the contact point has translational speed of zero



Kinetic Energy of Rolling Object



$$K = \frac{1}{2} I_0 \omega^2$$

$$= \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

$$= \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v^2$$

parallel axis theorem

rotational component

translational component