Chapter 7

Energy of a System

Scalar Product of Two Vectors

- Scalar product = dot product
- It's defined for two vectors of the same dimension
  - coordinate independent definition:

$$\vec{A}\vec{B} = AB\cos\theta$$

  - coordinate related definition (can be generalized for any number of dimensions):

$$(a_x\vec{i} + a_y\vec{j})(b_x\vec{i} + b_y\vec{j}) = a_x b_x + a_y b_y$$

- Commutative: $\vec{A}\vec{B} = \vec{B}\vec{A}$

$$\vec{A}\vec{A} = A^2$$

- Distributive: $\vec{A}(\vec{B}+\vec{C}) = \vec{A}\vec{B} + \vec{A}\vec{C}$

Work done by a Force

- Work done by force ***F*** over displacement Δ***r***:

$$W = \vec{F}\Delta\vec{r} \qquad\qquad [\mathrm{J}]=[\mathrm{N}][\mathrm{m}]$$

- Work done by a constant force: $W = \vec{F}\vec{r}$
- Work done by a force changing with ***r***: $W = \int \vec{F}\, d\vec{r}$
- Examples:
  - gravitational force $F=-mg$
  - spring force $F=-kx$

Kinetic Energy

- Combine definition of work and Newton's second law:

$$W = \int F\,dx = \int ma\,dx = \int m\frac{dv}{dt}dx = \int m\frac{dv}{dx}\frac{dx}{dt}dx =$$

$$\int m\frac{dv}{dx}v\,dx = \int mv\,dv = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = \Delta K$$

$$\text{Kinetic energy: } K = \frac{mv^2}{2}$$

Potential Energy

- Depending on particular force $\boldsymbol{F}$, one can find a scalar quantity U (which is a function of position $\boldsymbol{r}$) such that work done by $\boldsymbol{F}$ while moving from $\boldsymbol{r}_f$ to $\boldsymbol{r}_i$ is equal to a negative of the change of U

$$W = -\Delta U$$

Examples of Potential Energy

- Gravitational force:

$$W = -\int mg\,dy = -(mgy_f - mgy_i) = -(U_f - U_i)$$

$$U = mgy$$

- Spring force:

$$W = -\int kx\,dx = -\left(\frac{kx_f^2}{2} - \frac{kx_i^2}{2}\right) = -(U_f - U_i)$$

$$U = \frac{kx^2}{2}$$

Friction force: there is no associated potential energy

Conservative Forces

- Forces for those U exists are called conservative
- For a conservative force, work done by the force while moving the object from one point to another does not depend on the path (it only depends on the final and initial position of the object)
  - friction force is not like this
- Work done by a conservative force over a closed path is zero

Conservative Force and Potential Energy

$$W = \int F\,dx = -(U_f - U_i)$$

If we fix initial point (e.g. at origin), then

$$\int F\,dx = -(U(x) - U_i)$$

$$F = -\frac{dU}{dx}$$

Example: spring force

$$U = \frac{kx^2}{2}$$

$$F = -\frac{dU}{dx} = -kx$$

Equilibrium

- A particle is in mechanical equilibrium if net force acting on it is zero ($dU/dx=0$)
  - definition is more complicated for a system of particles
- Equilibrium is stable if, once a particle moves away from the point of equilibrium, a force is created which pushes the particle back to equilibrium
  - Examples: a mass on a spring ; a pendulum; freely falling particle at terminal speed
  - Condition for stable equilibrium: U has a minimum
- If created force pushes the particle further away from the point of equilibrium, equilibrium is unstable
  - Condition for unstable equilibrium: U has a maximum
- If U is flat within some region, the equilibrium is neutral