Physics 3513, Final exam: solution

(1) Let $z = x + iy$, then we have $x^2 - y^2 + 2xyi = x - iy$, or $x^2 - y^2 = x, 2xy = -y$. The second equation has two solutions $y = 0$ and $x = -1/2$. Substituting them to the first equation, we get $x = 0, 1$ for the first case and $y = \pm\sqrt{3}/2$ for the second one. **Answer:** $0, 1, (-1 \pm \sqrt{3})/2$.

(∗) $z + iz = (x - y) + (y + x)i$. Since $|4z^4|$ is real, this means $y + x = 0$, or $z = x(1 - i)$. Substituting, we get $2x = 16x^4$, so $x = 0$ or $x = 1/2$. **Answer:** $0, (1 - i)/2$.

(2) $g(\omega) = \frac{1}{2\pi}\int_{-a}^{+a} \frac{1}{2a} e^{-i\omega x}\, dx = \frac{1}{2\pi(-2ai\omega)}\left(e^{-i\omega a} - e^{i\omega a}\right) = \frac{\sin a\omega}{2\pi a\omega}.$

(∗) $\lim_{a\to 0} f(x) = \delta(x)$, so $\lim_{a\to 0}\int_{-\infty}^{+\infty} f(x)e^{-x^2}\, dx = 1$.

(3) $A_1\mathbf{e} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\mathbf{e}$, so $\lambda_1 = -1$. Similarly,

$A_2\mathbf{e} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2\mathbf{e}$, so $\lambda_2 = 2$.

(∗) $(A_1 + A_2)\mathbf{e} = A_1\mathbf{e} + A_2\mathbf{e} = -\mathbf{e} + 2\mathbf{e} = \mathbf{e}$, so $(A_1 + A_2)^{10}\mathbf{e} = (A_1 + A_2)^9\mathbf{e} = \cdots = \mathbf{e}$. The corresponding eigenvalue is 1.

(4) The characteristic equation $D^2 - 1 = 0$ has roots ± 1. Trying particular solutions of the form $k_1 e^{2x}$ and k_2, we find $k_1 = 1/3$ and $k_2 = 1$, respectively. **Answer:** $y = C_1 e^x + C_2 e^{-x} + \frac{1}{3}e^{2x} + 1$.

(∗) The equation can be re-written as $(y'' - y + 1)^2 = e^{2x}$, or $y'' - y = \pm e^x - 1$. **Answer:** $y = C_1 e^x + C_2 e^{-x} \pm \frac{1}{2}xe^x + 1$.

(5) The series converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1$, and diverges if $|x| > 1$. At $|x| = 1$, the series $\sum_{n=1}^{\infty}\frac{2^n + 1}{2^n n} > \sum_{n=1}^{\infty}\frac{1}{n}$ diverges, and the series $\sum_{n=1}^{\infty}\frac{2^n + 1}{2^n n}(-1)^n = \sum_{n=1}^{\infty}\frac{1 + 2^{-n}}{n}(-1)^n$ converges. Note that the original series can be split in two: $S(x) = \sum_{n=1}^{\infty}\frac{1}{n}x^n + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x}{2}\right)^n$ where the first part converges if $-1 \le x < 1$, and the second part converges if $-2 \le x < 2$. **Answer:**$-1 \le x < 1$.

(∗) $S(-1) = -\left(\ln(1 - x) + \ln\left(1 - \frac{x}{2}\right)\right)\Big|_{x=-1} = -\ln 2 - \ln\frac{3}{2} = -\ln 3.$

(6) $p = \frac{RT}{V} \to \left(\frac{\partial p}{\partial V}\right)_T = -\frac{RT}{V^2}$; $V = \frac{RT}{p} \to \left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{p}$; $T = \frac{pV}{R} \to \left(\frac{\partial T}{\partial p}\right)_V = \frac{V}{R}$. Multiplying the right-hand sides, we get $-\frac{RT}{V^2}\frac{R}{p}\frac{V}{R} = -\frac{RT}{pV} = -1$.

(∗) Let $x = x(y, z)$, then $\left(\frac{\partial F}{\partial y}\right)_z = F_x\left(\frac{\partial x}{\partial y}\right)_z + F_y = 0 \to \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}$, similarly $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}$ and $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$. Multiplying the right-hand sides, we get -1.